MMP Learning Seminar

Week 97.

Content:

Minimal log discrepancies of regularity one.

Minimal log discrepancy:

$(X, \Delta; x)$.

$$\mathrm{mld}(X, \Delta; x) = \min \{ a_E(X, \Delta; x) \mid c_X(E) = x \}$$

Regularity: $(X, \Gamma; x)$ be a log canonical sing.

$$\mathrm{reg}(X, \Gamma; x) = \dim \mathcal{D}(Y, \Gamma_Y), \text{ where } (Y, \Gamma_Y) \longrightarrow (X, \Gamma)$$

is a dlt modification.

Example: An elliptic sing has reg 0.

A klt sing has reg $-\infty$, $\dim \emptyset = -\infty$.

Absolute regularity: $(X, \Delta; x)$ be a klt singularity.

$$\hat{\mathrm{reg}}(X, \Delta; x) = \max \{ \mathrm{reg}(X, \Gamma; x) \mid (X, \Gamma; x) \text{ is lc } \& \ \Gamma \geq \Delta \}.$$

Examples:

$$\underbrace{A_n \xrightarrow{2:1} D_n,}_{\text{reg } 1.} \qquad \underbrace{E_6, E_7, E_8.}_{\text{reg} = 0}$$

(exceptional sing).

Some results towards ACC.

Here, we know ACC for mlds.

Theorem 0 (M-2021): Let n be a positive integer. There exists $\varepsilon_n > 0$ s.t ACC for mld's of regularity one & dim n holds in $[0, \varepsilon_n)$.

Theorem 0' (M-2021): Let n be a positive integer. Then there exists an upper bound for the mlds of regularity one & dim n.

Birational models & regularity:

Lemma 0: Let (X, B) & (X', B') be two crepant log CY pairs. Then $\operatorname{reg}(X,B) = \operatorname{reg}(X',B')$

$$\begin{tikzcd} Y \arrow[d, "p"'] \arrow[dr, bend left, "q"] & \\ X \arrow[r, dashed] & X' \end{tikzcd}$$

$$p^*(K_X + B) = q^*(K_{X'} + B').$$

A vanishing theorem:

Theorem 1 **(Ambro, 2006):** Let $(X, \sum_i b_i E_i)$ be a snc pair. $b_i \in [0,1]$. $X \xrightarrow{f} S$ a proper morphism, L Cartier on X s.t

$L - K_X - \sum_i b_i E_i$ is f-semiample

Let $q \geq 0$ & s be a local section of $R^q f_* \mathcal{O}_X(L)$ which is zero at the generic point of $f(X)$ & $f(C)$ for every lcc C of (X, B).

Then $s = 0$.

Proof: We may assume S affine & $f(X) = S$. Then

$\underbrace{L - K_X - \sum_i b_i E_i}_{\sum_i c_i E_j}$ is semiample. We may assume $L \sim_{\mathbb{Q}} K_X + \sum b_i E_i$.

f^*A does not contain lcc. & $R^q f_* \mathcal{O}_X(L) \longrightarrow R^q f_* \mathcal{O}_X(L) \otimes \mathcal{O}_S(A)$ is not injective

We may compactify X & S and assume

$$H^0(S, R^q f_* \mathcal{O}_X(L + f^*A)) \longrightarrow H^0(S, R^q f_* \mathcal{O}_X(L + 2f^*A))$$

is not injective

Consider the following commutative diagram:

$$
\begin{tikzcd}
E_2^{p,q} = H^p(S, R^q f_* \mathcal{O}_X(L+f^*A)) \arrow[r, Rightarrow] \arrow[d] & H^{p+q}(X, \mathcal{O}_X(L+f^*A)) \arrow[d] \\
\overline{E}_2^{p,q} = H^p(S, R^q f_* \mathcal{O}_X(L+2f^*A)) \arrow[r, Rightarrow] & H^{p+q}(X, \mathcal{O}_X(L+2f^*A))
\end{tikzcd}
$$

The map $E_2^{0,q} \longrightarrow H^q(X, \mathcal{O}_X(L+f^*A))$ is injective

So, we conclude that:

$$H^q(X, \mathcal{O}_X(L+f^*A)) \longrightarrow H^q(X, \mathcal{O}_X(L+2f^*A))$$

is non-injective. This contradicts Kollar's inj Thm. $\square$

Properties of log canonical centers:

Remark 1: $(X, \sum_{i\in I} E_i)$ snc pair, $I', I'' \subseteq I$ non-empty.
C a connected component of $\bigcap_{i\in I'} E_i$. Then
$C \subseteq \bigcup_{i\in I''} E_i$ if & only if $I' \cap I'' \neq \emptyset$.

Lemma 2: (X, B) log canonical pair. W union of lcc's.
$\mu: (X', B') \longrightarrow (X, B)$ log resolution s.t $\mu^{-1} W$ is a divisor.
& $\mu^{-1} W \cup \mathrm{supp}(B')$ is snc. Let S be the union of the divisors E on X' with $\mathrm{mult}_E(B') = 1$ & $E \subseteq \mu^{-1} W$.
Then $\mu_* \mathcal{O}_S = \mathcal{O}_W$.

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Then $\mu_* \mathcal{O}_S = \mathcal{O}_W$. $(K_{X'}+B' = \mu^*(K_X+B))$.

Proof:

$$\underbrace{B'}_{} = \underbrace{S}_{\text{contained } \mu^{-1}W} + \underbrace{R}_{\substack{\text{components of} \\ \text{coeff } 1 \text{ not contained } \mu^{-1}W}} + \underbrace{\Delta}_{\text{components with coeff in } (0,1)} - \underbrace{A}_{\substack{\text{comp with} \\ \text{neg coeff}}}.$$

No connected comp of int of components of R maps inside W. Remark 1.

Consider:

$$\mu_* \mathcal{O}_{X'}(\lceil A \rceil) \longrightarrow \mu_* \mathcal{O}_S(\lceil A|_S \rceil) \xrightarrow{0 \text{ map}} R^1\mu_* \mathcal{O}_{X'}(\lceil A \rceil - S).$$

By $W = \mu(S)$, $\lceil A \rceil - S = K_{X'} + R + \lceil A \rceil - A + \Delta - \underbrace{\mu^*(K_X+B)}_{\text{trivial over } X}$

By Theorem 1, the second map is the zero map.
We conclude $\mu_* \mathcal{O}_{X'}(\lceil A \rceil) \longrightarrow \mu_* \mathcal{O}_S(\lceil A|_S \rceil)$ is surjective.
Since $B \geq 0$, then A is μ-exc. Therefore

$$\mathcal{O}_X = \mu_* \mathcal{O}_S.$$

$\square$

Theorem 2: Let (X, B) be a log pair. W_1 & W_2 be two lcc.

If $W_1 \cap W_2 \neq \emptyset$, then $W_1 \cap W_2$ is union of lcc.

Proof: $x \in W_1 \cap W_2$, apply the Lemma to $W = W_1 \cup W_2$

$\mu : (X', B') \longrightarrow (X, B)$. E_1 E_2

S are all comp with coeff 1 mapping inside W, then.

$\mu_* \mathcal{O}_S = \mathcal{O}_W$, in particular. $S \longrightarrow W_1 \cup W_2$

has connected fibers. $x \in W_1 \cap W_2$.

$\pi^{-1}(x) \cap E_1 \cap E_2 \neq \emptyset$.

Blow-up $E_1 \cap E_2$ and obtain $E_3 \longrightarrow C_3 \subseteq X$

$x \in C_3 \subseteq W_1 \cap W_2$

$\square$

Structure Theorem for lCY pairs of reg 0:

Theorem (Kollár–Kovács, 2010): (X,B) log CY of reg 0.

Then, $\mathcal{D}(X,B)$ is either a point or two points.

If $\mathcal{D}(X,B)$ is disconnected, then there exists a bir modification

$$\begin{array}{ccc} (X,B) \dashrightarrow & (X',B') & \text{such that the general fiber is } \mathbb{P}^1 \\ & \downarrow & \& \ \lfloor B' \rfloor \text{ has two components dominating } Z. \\ & Z & \end{array}$$

Proof: Assume (X,B) is dlt. $S_1,\ldots,S_n \subseteq \lfloor B \rfloor$ are disc.

Run a MMP for $(X, B-\epsilon S_1)$.

$$\begin{array}{ccccc} (X_0,B_0) \dashrightarrow & (X_1,B_1) \dashrightarrow & \ldots \dashrightarrow & (X_n,B_n) & S_1 \text{ is ample over } Z. \\ \| & & & \downarrow & \\ (X,B) & & & Z. & \end{array}$$

For each i, the strict transforms of S_j's in X_i are disjoint.

Furthermore, every step of this MMP is S_1-positive.

- There is at least a second S_i. the dim of general fiber has dim 1. then it must be $\mathbb{P}^1$ & S_2 must be a 2nd section over Z.

- If there is a unique S_i, then we cannot conclude anything about $X_n \to Z$. Apply BAB to general fiber. $\square$

Curves on log CY pairs:

Corollary (Birkar-Kollár-Kovács, 2016): In the context of the previous Theorem assume further that $N(K_X+B)\sim 0$. Then, we have that:

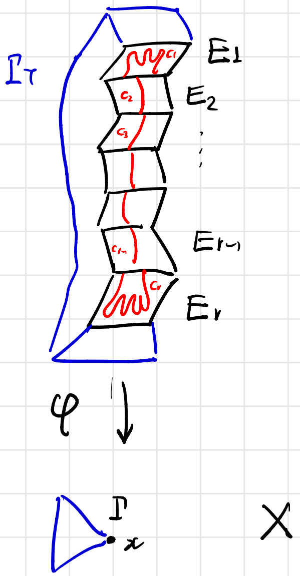
- If $\mathcal{D}(X,B)=pt$, then there exists a curve in the smooth locus $C\subseteq X^{sm}$ s.t $B.C\leq k(N,\dim X)$.
- If $\mathcal{D}(X,B)=\{p,q\}$, then there exists a curve in the smooth locus $C\subseteq X^{sm}$ s.t C only intersects $\lfloor B\rfloor$ twice & does not intersect $B-\lfloor B\rfloor$.

Proof of Thm 0:

$(X; x)$ be a klt sing of reg $= 1$.

$(X, \Gamma; x)$ bounded N-complement of reg $= 1$.

$(Y, \Gamma_Y + E_1 + \dots + E_r)$ its dlt modification.



If we perform adjunction to the exceptionals then we get CY pairs of reg 0.

$$\varphi^*(K_X + \Gamma) = K_Y + \Gamma_Y + E_1 + \dots + E_r.$$

$$\varphi^*(K_X) = K_Y + \Gamma_Y + (1-\alpha_1)E_1 + \dots + (1-\alpha_r)E_r.$$

$$\sum_i \alpha_i E_i + \Gamma_Y \sim_{\mathbb{Q},X} 0$$

we want to understand these numbers.

$$\sum_i \alpha_i (E_i \cdot C_j) + \Gamma_Y \cdot C_j = 0.$$

$C_1 . E_1 = -m_1.$ $\qquad$ $C_r \cdot E_r = -m_r$

$C_1 . E_2 = k_1.$ $\qquad$ $C_r \cdot E_{r-1} = k_r$

$C_1 \cdot \Gamma_Y = f_1$ $\qquad$ $C_r \cdot \Gamma_Y = f_2$

$$C_i \cdot E_{i-1} = 1$$

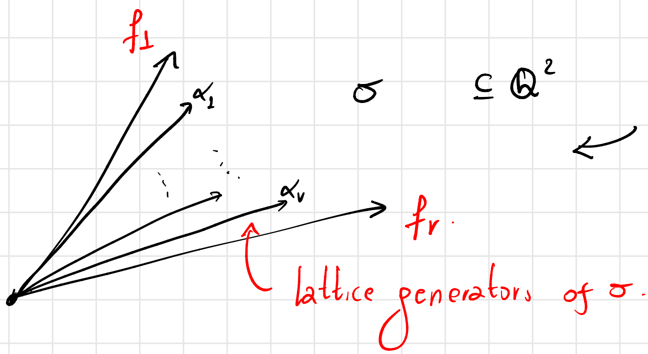
$$C_i \cdot E_{i+1} = 1.$$

$$C_i \cdot E_i = -m_i.$$

what is the minimum?

$$\begin{bmatrix} -m_1 & k_1 & 0 & \cdots & & 0 \\ 1 & -m_2 & 1 & & & \\ & 1 & -m_3 & 1 & & 0 \\ & & & \ddots & & \\ 0 & & & 1 & -m_{r-1} & 1 \\ & & & & k_r & -m_r \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_r \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \\ \vdots \\ 0 \\ f_r \end{bmatrix}$$

Find a cone in $\mathbb{Q}^2$ & a linear function. L



$\sigma \subseteq \mathbb{Q}^2$

satisfies ACC due to the surface toric case

$\square$

Remark: The expectation is that

mlds of reg $= r$ in an interval near 0.

behave as mlds of $(r+1)$-dim toric sing.